# Managing Informal Mathematical Knowledge: Techniques from Informal Logic

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# Outline

# Some Types of Informal Mathematical Knowledge

- Historical mathematics, falling short of modern rigour;
- High-level communication;
- Communication of work-in-progress between collaborators;
- Mathematics that is ‘respectable without being flawless’.

# 'A Foray into the Unknown'

I used to dislike intensely, *but have come to appreciate and even search for* . . . the situation where one has two, watertight well-designed arguments that lead inexorably to opposite conclusions. . . . Remember that research in mathematics involves a foray into the unknown. We may not know which of the two conclusions is correct or even have any feeling or guess. Proof at this point is our only arbiter. And it seems to have let us down. I have known myself to be in this situation for months on end. It induces obsessive and anti-social behaviour. Perhaps we have found an inconsistency in mathematics. But no, eventually a crack is found in one of the arguments and it begins to look more and more shaky. Eventually we kick ourselves for being so utterly stupid and life goes on. But it was no tool of logic that saved us. The search for a chink in the armour often involved many tricks including elaborate thought experiments and perhaps computer calculations. Much structural understanding is created, which is why I now so value this process. One's feeling of having obtained truth at the end is approaching the absolute. Though I should add that I have been forced to reverse the conclusion on occasions

Vaughan Jones, 'A credo of sorts.' In H. G. Dales, G. Oliveri, eds., *Truth in Mathematics*, Clarendon, Oxford, 1998, pp. 208 f.

# Lakatos's Method of Proofs and Refutations

Rule 1. If you have a conjecture, set out to prove it and to refute it. Inspect the proof carefully to prepare a list of non-trivial lemmas (proof-analysis); find counterexamples both to the conjecture (global counterexamples) and to the suspect lemmas (local counterexamples).

Rule 2. If you have a global counterexample discard your conjecture, add to your proof-analysis a suitable lemma that will be refuted by the counterexample, and replace the discarded conjecture by an improved one that incorporates the lemma as a condition. Do not allow a refutation to be dismissed as a monster. Try to make all 'hidden lemmas' explicit.

Rule 3. If you have a local counterexample, check to see whether it is also a global counterexample. If it is you can easily apply Rule 2.

Rule 4. If you have a counterexample which is local but not global, try to improve your proof analysis by replacing the refuted lemma by an unfalsified one.

Rule 5. If you have counterexamples of any type, try to find, by deductive guessing, a deeper theorem to which they are counterexamples no longer.

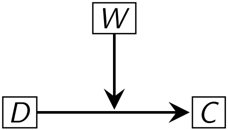
Imre Lakatos, *Proofs and Refutations: The Logic of Mathematical Discovery*, Cambridge University Press, Cambridge, 1976, pp. 50, 58, 76.

# What is Informal Logic?

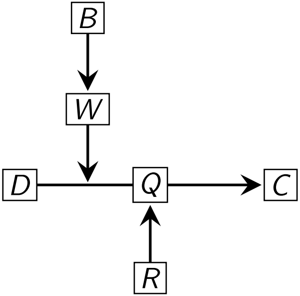
To speak of informal logic is not to contradict oneself but to acknowledge what should be obvious: that the understanding of natural arguments requires substantive knowledge and insights not captured in the axiomatized rules of formal logic.

Trudy Govier, *Problems in Argument Analysis and Evaluation*, Foris, Dordrecht, 1987, p. 204.

# What is a Toulmin Layout?



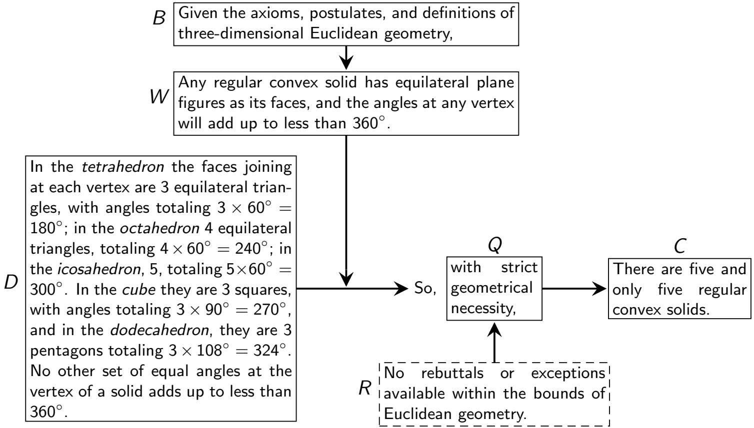

(a) Basic Layout $\langle D, W, C\rangle$




(b) Enhanced Layout $\langle D, W\langle B, Q, R\rangle, C\rangle$

# A Mathematical Example

## Theaetetus's proof that the platonic solids are exactly five in number




(Adapted from Toulmin *& al.*, *An Introduction to Reasoning*, Macmillan, New York, NY, 1979, Fig. 7.4, p. 89).

# The Pros and Cons of Visual Presentation

## Pro:

- Facilitates communication of complex ideas.

## Con:

- Time-consuming to produce.
- Frustrating to update or integrate with other systems.

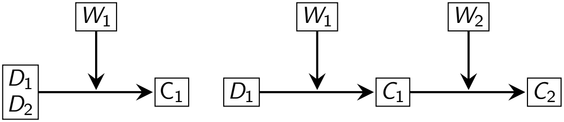
## Responses:

- Software automation of argument diagraming, *e.g. Araucaria*.
- Alternative use of non-graphical notation.

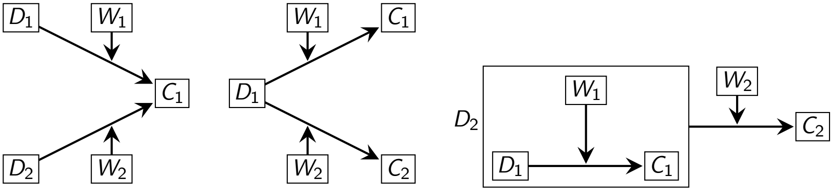
# Four Principles for Combining Layouts

I. Treat data and claim as the nodes in a graph or network.
II. Allow nodes to contain multiple propositions.
III. Any node may function as the data or claim of a new layout.
IV. The whole network may be treated as data in a new layout.

# Five Ways of Combining Layouts

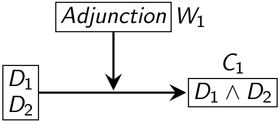



(a) Linked (b) Sequential

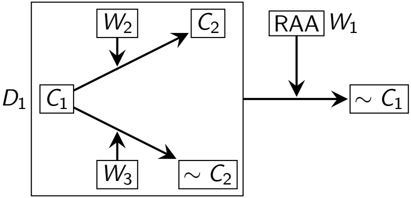



(c) Convergent (d) Divergent (e) Embedded

# Some common proof methods I

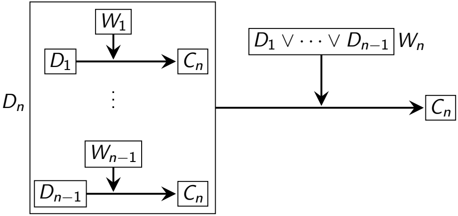



(a) Adjunction

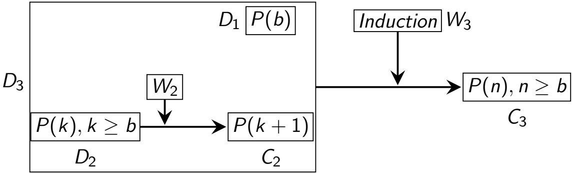



(b) Proof by Contradiction

# Some common proof methods II




(c) Proof by Cases




(d) (Weak) Induction

# Folding Compound Layouts into a Single Layout

## Without Embedding:

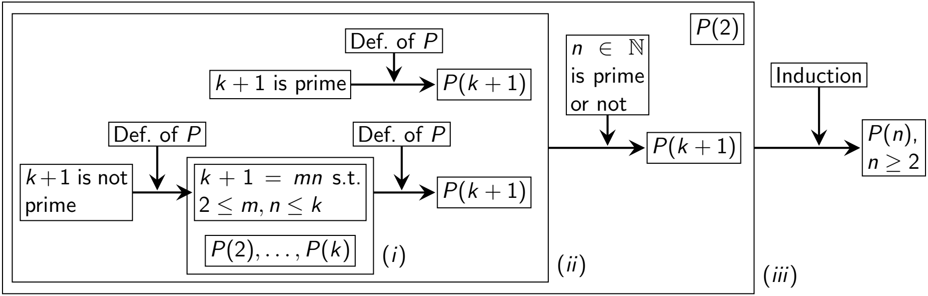
$$\left\langle \bigcup_{\text{in}(D_i)=0} D_i, \bigwedge_i W_i, \bigwedge_{\text{out}(C_i)=0} C_i \right\rangle \tag{1}$$

## Embedded Layouts:

$$\left\langle \bigcup_j (D_j \Rightarrow C_j), W_k \wedge \bigwedge_j W_j, C_k \right\rangle \tag{2}$$

# An Extended Example:

Proof that every natural number greater than one has a prime factorization.




($P(n)$ abbreviates '$n$ has a prime factorization'.)

$$\langle\{P(2), \{k+1 \text{ is prime} \Rightarrow P(k+1),$$
$$\{k+1 \text{ is not prime}, P(2), \ldots, P(k)\} \Rightarrow P(k+1)\} \Rightarrow P(k+1)\},$$
$$\text{Def. of } P \wedge n \in \mathbb{N} \text{ is prime or not} \wedge \text{Induction}, P(n) \text{ for } n \geq 2\rangle. \quad (3)$$

# Folding Enhanced Layouts into a Single Layout

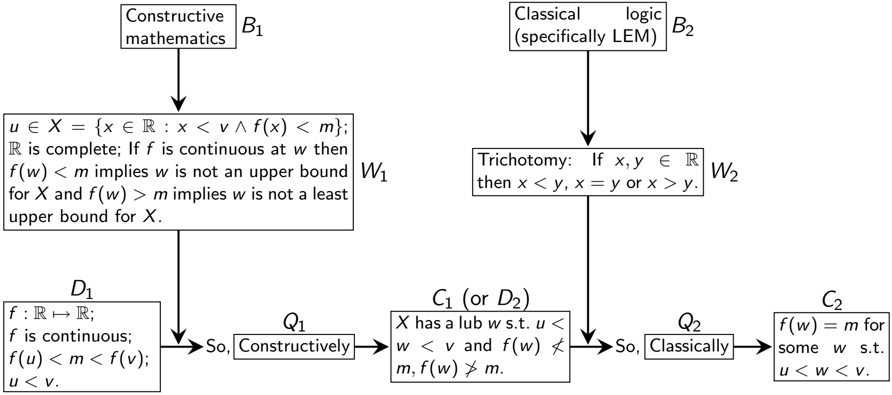
## Without Embedding:

$$\left\langle \bigcup_{\mathrm{in}(D_i)=0} D_i, \bigwedge_i W_i \left\langle \bigwedge_i B_i, \mathrm{lub} \bigcup_i Q_i, \bigvee_i R_i \right\rangle, \bigwedge_{\mathrm{out}(C_i)=0} C_i \right\rangle \quad (4)$$

## Embedded Layouts:

$$\left\langle \bigcup_j (D_j \Rightarrow_{Q_j} C_j), W_k \wedge \bigwedge_j W_j \left\langle B_k \wedge \bigwedge_j B_j, \mathrm{lub}\{Q_k \cup \bigcup_j Q_j\}, R_k \vee \bigvee_j R_j \right\rangle, C_k \right\rangle \quad (5)$$

# Classical Proof of the Intermediate Value Theorem




Classical proof of the Intermediate Value Theorem

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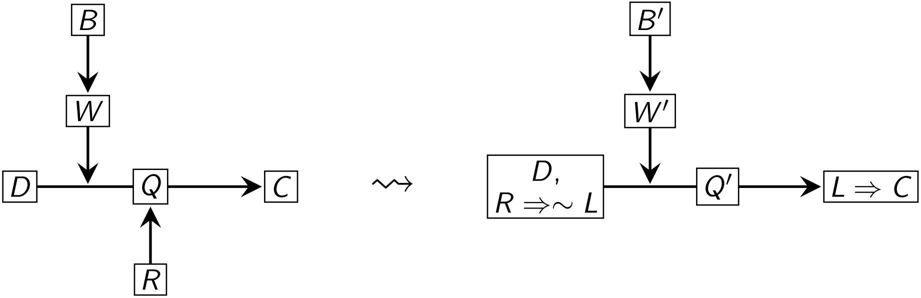
# Rebuttals




Lemma incorporation

# Summary

- The *proposed problem*: informal mathematical knowledge, and its management.
- The *proposed response*: make use of existing body of work in informal logic.
- *Specific example*: application of Toulmin layouts to mathematical inference.